Highly Precise Analytic Solutions for Classic Problem of Projectile Motion in the Air

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Abstract. Here is studied a classic problem of the motion of a projectile thrown at an angle to the horizon. The air drag force is taken into account as the quadratic resistance law. An analytic approach is used for the investigation. Equations of the projectile motion are solved analytically. All the basic functional dependencies of the problem are described by elementary functions. The found analytical solutions are highly accurate over a wide range of parameters. The motion of a baseball and a badminton shuttlecock are presented as examples.

The problem of the motion of a projectile in midair arouses interest of authors as before [1–3]. Together with the investigation of the problem by numerical methods, attempts are still being made to obtain the analytical solutions All proposed approximate analytical solutions are rather complicated and inconvenient for educational purposes. This is why the description of the projectile motion by means of a simple approximate analytical formulas under the quadratic air resistance is of great methodological and educational importance. The purpose of the present work is to give a simple formulas for the construction of the trajectory of the projectile motion with quadratic air resistance. It allows to construct a trajectory of the projectile with the help of elementary functions without using numerical schemes. The drag force will be written as $R = mgkV^2$. The well-known solution of differential equations of the projectile motion consists of an explicit analytical dependence of the velocity on the slope angle of the trajectory and three quadratures [2]

$$V(\theta) = \frac{V_0 \cos\theta_0}{\cos\theta\sqrt{1 + kV_0^2\cos^2\theta_0\left(f(\theta_0) - f(\theta)\right)}}, \quad f(\theta) = \frac{\sin\theta}{\cos^2\theta} + \ln tg\left(\frac{\theta}{2} + \frac{\pi}{4}\right), \tag{1}$$

$$x = x_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 d\theta, \quad y = y_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 tg\theta d\theta, \quad t = t_0 - \frac{1}{g}\int_{\theta_0}^{\theta} \frac{V}{cos\theta} d\theta . \tag{2}$$

The integrals on the right-hand sides of formulas (2) cannot be expressed in terms of elementary functions. Hence, to determine the variables *t, x* and *y* we must evaluate the definite integrals (2). The task analysis shows, that equations (2) are not exactly integrable owing to the complicated nature of function $f(\theta)$ in formulas (1). Therefore, it can be assumed that a successful approximation of this function will make it possible to calculate analytically the definite integrals (2) with the required accuracy. An analysis of the problem shows that it is convenient to approximate the function $f(\theta)$ only by polynomials of the second or third degree. Approximation of the function $f(\theta)$ by a second order polynomial $f_2(\theta)$ has the following form

$$f_2 \ \theta \ = \begin{cases} a_1\tan\theta + b_1\tan^2\theta, & \text{on} \quad \text{condition} \quad \theta \geq 0, \\ a_1\tan\theta - b_1\tan^2\theta, & \text{on} \quad \text{condition} \quad \theta \leq 0. \end{cases}$$

The coefficients a_1 and b_1 can be chosen in such a way as to smoothly connect the functions $f(\theta)$ and $f_2(\theta)$ to each other with the help of conditions $f_2(\theta_0) = f(\theta_0)$, $f_2'(\theta_0) = f'(\theta_0)$. Approximation of the function $f(\theta)$ by a third-order polynomial $f_3(\theta)$ has the following form $f_3(\theta) = a_1 \tan\theta + b_1 \tan^3\theta$. The function $f_3(\theta)$ is applicable over the whole interval $-0.5\pi < \theta < 0.5\pi$. Such a functions $f_2(\theta)$, $f_3(\theta)$ well approximate the function $f(\theta)$ throughout the whole interval of its definition for any values θ_0. Hence, the functions $f_2(\theta)$, $f_3(\theta)$ can be used instead of the function $f(\theta)$ in calculating the integrals (2). Now the quadratures (2) are integrated in elementary functions [2,3].

Obtained in [2,3] formulas have a wide region of application. We introduce the notation $p = kV_0^2$. The dimensionless parameter p has the following physical meaning – it is the ratio of air resistance to the weight of the projectile at the beginning of the movement. As calculations show, trajectory of the projectile $y = y(x)$ and the main characteristics of the motion have high accuracy for values of the launch angle and for the parameter p within ranges $0° < \theta_0 < 90°$, $0 < p \le 60$. Figure 1 presents the results of plotting the projectile trajectories with the aid of obtained formulas over a wide range of the change of the initial angle θ_0

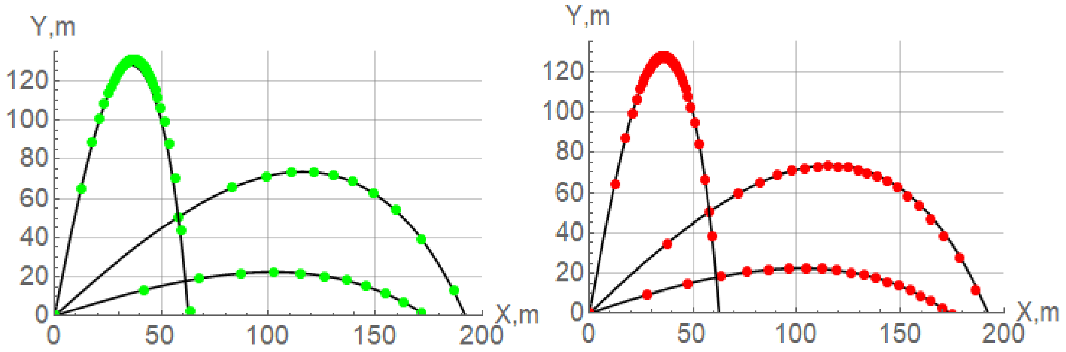

Figure 1 The graphs of the trajectory $y = y(x)$ at launching angles $\theta_0 = 20°, 45°, 80°$

Analytical solutions are shown in Fig. 1 by dotted lines. The thick solid lines in Fig. 1 are obtained by numerical integration of system (1) with the aid of the 4-th order Runge-Kutta method. The red dots lines are obtained with using approximation $f_2(\theta)$. The green dots lines are obtained with using approximation $f_3(\theta)$. As it can be seen from Fig. 1 the analytical solutions (dotted lines) and a numerical solutions are the same. It is interesting that identical trajectories are described with various analytical formulas. Thus, a successful approximation of the function $f(\theta)$ made it possible to calculate the integrals (2) in elementary functions and to obtain a highly accurate analytical solution of the problem of the motion of the projectile in the air.

References

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