Why are complex numbers useful in quantum mechanics? Looking for justifications in physics textbooks

Ricardo KARAM

Department of Science Education, University of Copenhagen, Denmark

Abstract. Complex numbers are broadly used in physics, usually as a calculation tool that makes things easier due to Euler's formula. In the end, it is only the real component that has physical meaning or the two parts (real and imaginary) are treated separately as real quantities. However, the situation seems to be different in quantum mechanics, since the imaginary unit *i* appears explicitly in its fundamental equations. From a learning perspective, it is desirable that students reflect on the reasons why a particular mathematical structure is useful to describe a physical property. In this study, we investigate how/if justifications for the use of complex numbers are presented in quantum mechanics textbooks that are widely used at undergraduate level. The analysis is complemented by other possible justifications and some historical considerations.

1 Introduction

Complex numbers were invented[1] in 16^{th}-century Italy as a calculation tool to solve cubic equations [1]. In the beginning, not much attention was paid to their ontology, since in the end these numbers would "cancel out" and only (real) roots were considered. Around 250 years later, complex numbers were given a geometrical interpretation and, since then, they became quite a useful tool to physics [2], essentially because they represent direction algebraically (2D vectors) and many of their operations have a direct geometrical meaning (e.g., the product rule: "multiply the norms and add the angles").

It is particularly helpful to use complex numbers to model periodic phenomena, especially to deal with phase differences. Mathematically, one can treat a physical quantity as being complex, but address physical meaning only to its real part. Another alternative is to treat the real and imaginary parts of a complex numbers as two related (real) physical quantities.

The situation seems substantially different in quantum mechanics. It suffices to look at its fundamental equations, both in the matrix and wave formulations,

$$\boldsymbol{pq} - \boldsymbol{qp} = -i\hbar$$

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi$$

to wonder about the presence of the imaginary unit i [3]. Thus, a pedagogical question arises naturally: Is the use of complex numbers in quantum mechanics thematised or even justified in physics textbooks? Making such justifications explicit in teaching should encourage students to focus on the structural role of mathematics in physics [4].

2 Design

In this study, we will look for explicit justifications for the use of complex numbers in quantum mechanics textbooks, especially when these mathematical structures are presented for the first time. For instance, suppose that at some point it is mentioned that the wave function is complex; is there an explicit justification for that? Or, when vector states are said to be complex,

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[1] Or discovered, depending on your philosophical affiliation.

that operators need to be Hermitian or inner products are complex, is it possible to find some specific reason for the use of complex numbers? If so, what is the nature of these reasons?

The choice of the sample [5-10] is not only to make it somehow representative, but also due to the fact that these textbooks have rather different approaches to teach quantum mechanics. Thus, it is reasonable to expect that structural differences in the way the concepts are presented may lead to different ways to justify the usefulness of complex numbers in quantum mechanics.

3 Perspectives

Reasons for why complex numbers are needed (if at all) in quantum mechanics are widely discussed in online fora, which reveals that it seems a reasonable question to ask. However, a preliminary analysis of the selected textbooks shows that explicit justifications are scarce. It is usually assumed, implicitly, that there is no problem in using complex numbers and defining complex valued quantities, which essentially cannot be separated into (pure) real and imaginary parts.

At a first glance, there appears to be a possible distinction between kinds of justifications: i) the more formal ones, e.g., "In a complex vector space *every* linear transformation has eigenvectors" [6] (p. 87) and ii) the more "phenomenological" ones, e.g., "One might think one could measure a complex dynamical variable by measuring separately its real and pure imaginary parts. But this would involve two measurements or two observations, which would be alright in classical mechanics, but would not do in quantum mechanics, where two observations in general interfere with one another" [11] (p. 35). The next step of this investigation is to make a more systematic categorization and give concrete examples of the different justifications. Original justifications found in important historical papers, as well as other ways to justify found in the literature, shall complement this study.

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